4.3. Homogeneous Linear Equations with Constant Coefficients.

In this section we will study solution of the homogeneous linear equations

$$a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = 0$$

where $a_n(x), \dots, a_1(x), a_0(x)$ are constants $a_n, \dots, a_1, a_0$.

Let us start with the first-order eq

$$ay' + by = 0 \quad (1)$$

We can solve it by separating variables or using integrating factors. However, we can use the following algebraic method:

If we substitute $y = e^{mx}$, then $y' = me^{mx}$ and our eq (1) becomes:

$$ame^{mx} + be^{mx} = 0$$

$$\Leftrightarrow \quad (am+b)e^{x} = 0$$

$$\Leftrightarrow \quad am + b = 0$$

$\Rightarrow \quad y = e^{-b/a\,x}$ is a solution of (1).

And $\quad y = ce^{-b/a\,x}$ is the general solution of (1).

AUXILIARY EQUATION

Let us consider next the second-order eq:

$$ay'' + by' + cy = 0 \qquad (2).$$

If we try to find a solution of form $y = e^{mx}$, then after substituting

$$y' = me^{mx} \quad \text{and} \quad y'' = m^2 e^{mx}$$

to (2), we get

$$am^2 e^{mx} + bme^{mx} + ce^{mx} = 0$$

or

$$(am^2 + bm + c)e^{mx} = 0.$$

Thus, $y = e^{mx}$ is a solution of (2) when m satisfies

$$am^2 + bm + c = 0 \qquad (3).$$

The eq (3) is called the **auxiliary equation** of the DE (2).

There are 3 cases to distinguish:

<u>CASE I</u>: $b^2 - 4ac > 0$.

Then (3) has two distinct real roots

$$m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$$

and

$$m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Then the general solution of (2) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

CASE II: $b^2 = 4ae$

The auxiliary eq (3) has a repeated root $m = -\frac{b}{2a}$.
$\Rightarrow$ we have a solution $y_1 = e^{mx}$.

By reduction of order method in 4.2, we can find a second solution of (2) as

$$y_2 = e^{mx} \int \frac{e^{2mx}}{e^{2mx}} dx = e^{mx} \int dx = x e^{mx}.$$

The general solution is

$$y = c_1 e^{mx} + c_2 x e^{mx}.$$

CASE III: $b < 4ae$

The eq (3) has conjugate complex roots.
$m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$,
where α and $\beta > 0$ are real.

Thus $y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}$
is the general solution of eq (2). However, we would like to avoid complex number. We use Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

$$\Rightarrow \quad e^{i\beta x} = \cos\beta x + i\sin\beta x$$

$$\Rightarrow \quad e^{i\beta x} + e^{-i\beta x} = 2\cos\beta x, \quad e^{i\beta x} - e^{-i\beta x} = 2i\sin\beta x.$$

Let us choose $C_1 = C_2 = 1$,

$$Y_1 = e^{(\alpha+i\beta)x} + e^{(\alpha-i\beta)x},$$

and $C_1 = 1$ and $C_2 = -1$,

$$Y_2 = e^{(\alpha+i\beta)x} - e^{(\alpha-i\beta)x}.$$

Thus

$$Y_1 = e^{ax}\left(e^{i\beta x} + e^{-i\beta x}\right) = 2e^{ax}\cos\beta x$$

$$Y_2 = e^{ax}\left(e^{i\beta x} - e^{-i\beta x}\right) = 2i\, e^{ax}\sin\beta x.$$

$\Rightarrow$ $$\frac{1}{2}Y_1 = e^{ax}\cos\beta x$$

and $-\frac{i}{2}Y_2 = e^{ax}\sin\beta x$ are two real solutions of eq (2). Then the solution has form

$$y = C_1 e^{ax}\cos\beta x + C_2 e^{ax}\sin\beta x.$$

Example 1. Solve the following differential equations

$$\boxed{a)\ 2y'' - 5y' - 3y = 0}$$

$$2m^2 - 5m - 3 = (2m+1)(m-3) = 0$$

$$m_1 = -\frac{1}{2},\ m_2 = 3$$

$\Rightarrow$

$$y = C_1 e^{-\frac{x}{2}} + C_2 e^{3x}.$$

$$\boxed{b)\ y'' - 10y' + 25y = 0}$$

$$m^2 - 10m + 25 = (m-5)^2 = 0,\ m_1 = m_2 = m = 5$$

$\Rightarrow \quad y = C_1 e^{5x} + C_2 x e^{5x}.$

$\boxed{\text{c)} \quad y'' + 4y' + 7y = 0}$

$m^2 + 4m + 7 = 0, \quad m_1 = -2 + \sqrt{3}\, i$
$\qquad\qquad\qquad\qquad m_2 = -2 - \sqrt{3}\, i$

$\Rightarrow$

$y = c_1 e^{-2x} \cos\sqrt{3}\, x + C_2 e^{-2x} \sin\sqrt{3}\, x.$

<u>Example</u>: Solve the IVP:

$$4y'' + 4y' + 17y = 0$$
$$y(0) = -1, \; y'(0) = 2$$

<u>Solution</u>: Let $y = e^{mx}$, the auxiliary eq is

$$4m^2 + 4m + 17 = 0$$

There are two conjugate complex solotions

$$m_1 = -\frac{1}{2} + 2i \quad \text{and} \quad m_2 = -\frac{1}{2} - 2i.$$

$\Rightarrow$

$$y = C_1 e^{-x/2} \cos 2x + C_2 e^{-x/2} \sin 2x.$$

Apply $y(0) = -1$, we have

$$-1 = C_1 \cos 0 + C_2 \sin 0$$
$$\Leftrightarrow \quad -1 = C_1.$$

We have

$$y' = -(e^{-x/2} \cos 2x)' + C_2 (e^{-x/2} \sin 2x)'$$
$$= \left(e^{-x/2} \sin 2x + \frac{1}{2} e^{-x/2} \cos 2x\right)$$
$$+ C_2 \left(-\frac{1}{2} e^{x/2} \sin 2x + 2e^{-x/2} \cos 2x\right)$$

Apply $y'(0) = 2$:

$$2C_2 + \frac{1}{2} = 2 \Leftrightarrow \quad C_2 = 3/4.$$

Solution of the IVP is

$$y = e^{-x/2}\left(-\cos 2x + \frac{3}{4}\sin 2x\right).$$

TWO IMPORTANT EQUATIONS

① $y'' + k^2 y = 0$

② $y'' - k^2 y = 0$

By investigate the auxiliary equation $m^2 + k^2 = 0$, with two complex roots $m_1 = ki$ and $m_2 = -ki$.

$\Rightarrow$ Solution of (1) is

$$y = c_1 \cos kx + c_2 \sin kx$$

Similarly solution of (2) is

$$y = c_1 e^{kx} + c_2 e^{-kx}.$$

Note that:

$$\cosh kx = \frac{e^{kx} + e^{-kx}}{2}$$

and

$$\sinh kx = \frac{e^{kx} - e^{-kx}}{2}.$$

Then (2) has solution:

$$y = c_1 \cosh kx + c_2 \sinh kx.$$

HIGHER-ORDER EQUATIONS

In general the nth-order homogeneous eq:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

has the auxiliary eq:

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0$$

If this eq has all n real distinct real roots then the general solution has form:

$$y = c_1 e^{m_1 x} + \cdots + c_n e^{m_n x}.$$

However it is harder to sumerize the anologues of cases II and III.

If m_1 has multiplicity k, then the general solution most contain the linear combination

$$c_1 e^{m_1} + c_2 x e^{m_1 x} + c_3 x^2 e^{m_1 x} + \cdots + c_k x^{k-1} e^{m_1 x}.$$

Finally if the auxiliary eq has complex roots, then these roots come with conjugate pairs. That is if $m_1 = \alpha + \beta i$ is a root, then $m_2 = \alpha - \beta i$ is also a root and by Euler's identity

$$c_1 e^{(\alpha+\beta i)x} + c_2 e^{(\alpha-\beta i)x}$$

can be written as

$$c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x.$$

<u>Example</u>. Solve $y''' + 3y'' - 4y = 0$

The auxiliary eq is: $m^3 + 3m^2 - 4m = 0$ has roots $m_1 = 1$, $m_2 = m_3 = -2$.

The general solution is

$$y = c_1 e^x + c_2 e^{-2x} + c_3 x e^{-2x}.$$

Example: Solve
$$y^{(4)} + 2y'' + y = 0.$$
The auxiliary eq
$$m^2 + 2m^2 + 1 = (m^2+1)^2 = 0 \text{ has roots}$$
$m_1 = m_3 = i$ and $m_2 = m_4 = -i$.
The general solution has form
$$y = C_1 e^{ix} + C_2 e^{-ix} + C_3 x e^{ix} + C_4 x e^{-ix}$$
By Euler's identity
$$C_1 e^{ix} + C_2 e^{-ix} = c_1 \cos x + c_2 \sin x$$
$$C_3 x e^{ix} + C_4 x e^{ix} = c_3 x \cos x + c_4 x \sin x.$$
Thus the general solution has form
$$y = c_1 e^{ix} + c_2 e^{-ix} + c_3 x e^{ix} + c_4 x e^{-ix}. \quad \square$$

In general if the complex root $m_1 = \alpha + \beta i$ has multiplicity k, then its conjugate $m_2 = \alpha - \beta i$ is also a root of multiplicity k.
Then the general solution must contain the linear combination of $2k$ solutions
$$e^{\alpha x}\cos\beta x,\ x e^{\alpha x}\cos\beta x,\ \dots,\ x^{k-1} e^{\alpha x}\cos\beta x$$
$$e^{\alpha x}\sin\beta x,\ x e^{\alpha x}\sin\beta x,\ \dots,\ x^{k-1} e^{\alpha x}\sin\beta x.$$

Example: Solve $3y''' + 5y'' + 10y' - 4y = 0$

The auxiliary eq is $3m^3 + 5m^2 + 10m - 4 = 0$

We try to find a root of the eq.

+ If an integer root exists, it must be a factor of the last coefficient a_0. That is it must be $\pm 1, \pm 2$, or ± 4.

But it is not the case here.

+ We try to find a rational root of form

$$\frac{p}{q}, \quad \begin{array}{l} p \text{ is a divisor of } a_0 \\ q \text{ is a divisor of } a_3. \end{array}$$

$$\Rightarrow \frac{p}{q}: \pm 1, \pm 2, \pm 3, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3}$$

Checking each of the cases, we get a root $m_1 = \frac{1}{3}$

$$\Rightarrow 3m^2 + 5m^2 + 10m - 4 = \left(m - \frac{1}{3}\right)(3m^2 + 6m + 12)$$

The quadratic eq

$$3m^2 + 6m + 12 = 0$$

has two complex roots

$$m_2 = -1 - \sqrt{3}\, i, \quad m_2 = -1 + \sqrt{3}\, i$$

Therefore, the general solution is

$$y = c_1 e^{x/3} + c_2 e^{-x} \cos \sqrt{3}\, x + c_3 e^{-x} \sin \sqrt{3} x.$$